[13.31] Let ***V*** be a vector space and ***T*** be a linear transformation on ***V*** with distinct eigenvalues **1, …, **m, where *m* ≤ n. Furthermore assume

1. For each *j* of multiplicity  ≥ 2 (if any), there are  independent eigenvectors.

Prove (**A)** there is a basis for ***V*** composed of eigenvectors and **(B)**

T = 

where each *j* appears *rj* times.

**Solution**. **(A)**

Let  be the multiplicity of eigenvalue ***j*. Since there are *n* eigenvalues, we have that  Let **B** *j* =  be the set of  independent eigenvectors corresponding to *j*. We wish to prove 

comprises a basis for ***V***. Since **B**contains *n* vectors, it suffices to show that these vectors are linearly independent. So, assume



We will be done if we can show that  so suppose some  We show this leads to a contradiction, which will complete the proof.

Since the double sum (\*) has a finite number of terms, there is some collection  of non-zero coefficients satisfying (\*) having as few terms as possible. That is,  and a set 

such that

 has the minimum number of terms.

If *p* = 1, all the eigenvectors arise from a single eigenvalue and are thus independent by condition (a). Hence  contradicting that they are all non-zero.

So, we assume *p* > 1. Since we can apply *T* to equation (1) to get



Multiplying equation (1) by *p* gives



Subtracting (3) from (2) gives





In equation (4),  and the eigenvalues are distinct. Thus we have produced a shorter relation than (1), yielding the afore-mentioned contradiction, completing the proof of (A). ✔

**(B)** Re-label  and re-label the corresponding eigenvalues as . (For clarification, in this notation if, for example, **1 has multiplicity > 1, then we will have . In the basis   is written in matrix form as





That is,

T = 

 we revert to the original notation where each Eigenvalue appears according to its multiplicity, then this matrix becomes the required matrix. 